

Integrable Yang-Mills-Higgs Equations in 3-Dimensional De Sitter Space-Time.

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Abstract. This paper describes an integrable Yang-Mills-Higgs system on (2+1)-dimensional de Sitter space-time. It is the curved-space-time analogue of the Bogomolnyi equations for monopoles on \mathbf{R}^3 . A number of solutions, of various types, are constructed.

1. INTRODUCTION.

The background to this paper is the question of the existence of integrable nonlinear partial differential equations (and more specifically of soliton equations) in curved space-times. For a given (fixed) space-time M , are there integrable systems which live on M (*i.e.* are covariantly coupled to its geometry)? In general, this places severe restrictions both on M and on the equations that are coupled to it. In this paper, we concentrate on one example, namely an integrable Yang-Mills-Higgs system on (2+1)-dimensional de Sitter space-time.

In effect, this generalizes examples which have long been known. Consider the chiral equation $g^{\mu\nu}\partial_\mu(U^{-1}\partial_\nu U) = 0$, where $U(x^\mu)$ takes values in a Lie group, and where $g_{\mu\nu}$ is the metric of M . This system is integrable if M is (1+1)-dimensional (this being related to conformal invariance). In higher-dimensional flat space-times, the chiral equation is not integrable¹; and this is probably also the case for curved space-times of dimension greater than two. But if one modifies the equation by adding a torsion term, then integrability is possible^{1,2}; in particular, there is an integrable (modified) chiral equation in flat 3-dimensional space-time \mathbf{R}^{2+1} . The system is equivalent to one involving a gauge field (Yang-Mills field) coupled to a Higgs field, and may be seen to arise from the self-dual Yang-Mills equations in \mathbf{R}^{2+2} , by dimensional reduction. The soliton solutions can be understood in terms of algebraic geometry, and the soliton dynamics is (in general non-trivial³⁻¹⁰).

Other ways of reducing the self-dual Yang-Mills equations in \mathbf{R}^{2+2} can lead to integrable Yang-Mills-Higgs systems in curved (2+1)-dimensional space-times. These are the Lorentzian analogue of hyperbolic monopoles, which live on (positive-definite) hyperbolic 3-space. The space-time has to have constant curvature; and so there are two Lorentzian possibilities, namely anti-de Sitter and de Sitter space-time. Some preliminary results on the anti-de Sitter case have appeared previously¹¹; the present paper deals with the de Sitter case. In particular, we construct various explicit solutions. One new feature that appears here is associated with the non-trivial topology of de Sitter space.

2. (2+1)-DIMENSIONAL DE SITTER SPACE-TIME.

(2+1)-dimensional de Sitter space-time M is the manifold $\mathbf{R} \times S^2$ equipped with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \cosh^2 T (d\theta^2 + \sin^2 \theta d\varphi^2) - dT^2. \quad (1)$$

Here $T \in \mathbf{R}$ is a time coordinate, and (θ, φ) are polar coordinates on the spatial sphere. It is a space of constant curvature, with scalar curvature $R = 6$ (conventions are those of ref 12).

There is a relation between this space-time M and flat $(2+2)$ -dimensional space \mathbf{R}^{2+2} , and we shall use this to obtain equations on M from equations on \mathbf{R}^{2+2} . The relation is as follows. Let u and w be complex coordinates on \mathbf{R}^{2+2} , so that its metric is $du d\bar{u} - dw d\bar{w}$. First, define new coordinates $(\theta, \varphi, \tilde{\theta}, \tilde{\varphi})$ by

$$u = \frac{(\sin \theta)e^{-i\varphi}}{(\cos \theta + \cos \tilde{\theta})}, \quad w = \frac{(\sin \tilde{\theta})e^{i\tilde{\varphi}}}{(\cos \theta + \cos \tilde{\theta})}. \quad (2)$$

Then

$$du d\bar{u} - dw d\bar{w} = 2(\cos \theta + \cos \tilde{\theta})^{-1} ds_{\widetilde{M}}^2, \quad (3)$$

where

$$ds_{\widetilde{M}}^2 = (d\theta^2 + \sin^2 \theta d\varphi^2) - (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2). \quad (4)$$

In other words, R^{2+2} is conformal to part of the product $\widetilde{M} = S^2 \times S^2$; we interpret (θ, φ) as polar coordinates on the first sphere of \widetilde{M} , and $(\tilde{\theta}, \tilde{\varphi})$ on the second. The 4-space \widetilde{M} is conformally flat and has vanishing scalar curvature; it is a double cover of a conformal compactification^{13,14} of \mathbf{R}^{2+2} .

The next step is to reduce to 2+1 dimensions: this is done by factoring out by the Killing vector $\partial/\partial\tilde{\varphi}$, *i.e.* by a rotation of the second sphere. First we remove $\tilde{\theta} = 0$ and $\tilde{\theta} = \pi$, which are fixed points of the rotation. On the complement of these fixed points, we can write

$$ds_{\widetilde{M}}^2 = \sin^2 \tilde{\theta} [\operatorname{cosec}^2 \tilde{\theta} (d\theta^2 + \sin^2 \theta d\varphi^2 - d\tilde{\theta}^2) - d\tilde{\varphi}^2]. \quad (5)$$

So \widetilde{M} (minus the fixed points) is conformal to the product of S^1 and a space with topology $\mathbf{R} \times S^2$ and metric

$$ds^2 = \operatorname{cosec}^2 \tilde{\theta} (d\theta^2 + \sin^2 \theta d\varphi^2 - d\tilde{\theta}^2). \quad (6)$$

This is exactly $(2+1)$ -dimensional de Sitter space-time (1), where the coordinates T and $\tilde{\theta}$ are related by $\tanh T = -\cos \tilde{\theta}$.

3. INTEGRABLE EQUATIONS ON M .

In view of the conformal relation between \widetilde{M} and M , we may obtain integrable equations on M by reducing conformally-invariant integrable equations on \widetilde{M} (or \mathbf{R}^{2+2}). The simplest

conformally-invariant equation on \widetilde{M} is the conformally-invariant wave equation. Bearing in mind the absence of scalar curvature, this has the form

$$\Delta\chi - \widetilde{\Delta}\chi = 0, \quad (7)$$

where Δ and $\widetilde{\Delta}$ are the Laplacians on the two spheres. The $\tilde{\varphi}$ -independent solutions of (7) correspond to solutions of the conformally-invariant wave equation on M , namely

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi - \Psi = 0, \quad (8)$$

where χ and Ψ are related by the relevant conformal factor: $\Psi = (\text{sech } T)\chi$. Solutions can be obtained (in terms of Legendre polynomials and spherical harmonics) by separating variables or by twistor methods¹³. For example, the simplest case $\chi = 1$ (constant) gives $\Psi = \text{sech } T$, *i.e.* a solution of (8) which is spatially constant. Using $l = 1$ spherical harmonics yields the examples $\Psi = \text{sech } T \tanh T \cos \theta$, $\Psi = \text{sech } T \tanh T \sin \theta \cos \varphi$ etc.

Another example, and the one which we concentrate on in this article, is that of the self-dual Yang-Mills equations (these are integrable on any conformally-flat 4-space, and so in particular on \widetilde{M}). When we reduce to the (2+1)-dimensional space-time M , the self-dual Yang-Mills field becomes a Yang-Mills-Higgs system (Φ, A_μ) satisfying the Bogomolny-type equations

$$D_\alpha \Phi = \frac{1}{2}\eta_{\alpha\beta\gamma}F^{\beta\gamma}. \quad (9)$$

The Higgs field Φ (taking values in the Lie algebra G of the gauge group) is identified with the $\tilde{\varphi}$ -component $A_{\tilde{\varphi}}$ of the gauge potential, with the remaining three components A_μ becoming a gauge potential on M . As usual, D_α denotes the covariant derivative $D_\alpha\Phi = \partial_\alpha\Phi + [A_\alpha, \Phi]$, $F_{\mu\nu}$ is the gauge field $[D_\mu, D_\nu]$, and $\eta_{\alpha\beta\gamma} = [-\det(g_{\mu\nu})]^{1/2}\varepsilon_{\alpha\beta\gamma}$ is the volume 3-form on M . In terms of the polar coordinates $(\theta, \tilde{\theta}, \varphi)$, eqn (9) is

$$\begin{aligned} D_{\tilde{\theta}}\Phi &= (\sin \tilde{\theta} / \sin \theta) F_{\theta\varphi}, \\ D_\theta\Phi &= (\sin \tilde{\theta} / \sin \theta) F_{\tilde{\theta}\varphi}, \\ D_\varphi\Phi &= (\sin \tilde{\theta} \sin \theta) F_{\theta\tilde{\theta}}. \end{aligned} \quad (10)$$

So (9), or equivalently (10), form a set of covariant integrable partial differential equations on M . They are linear if the gauge algebra G is abelian, but otherwise are nonlinear. In the remaining sections, we shall construct and examine some solutions of (10), for gauge algebras $u(1)$ and $su(2)$.

4. A U(1) EXAMPLE.

For gauge algebra $G = u(1)$, the equations (10) reduce to

$$\partial_\alpha \Phi = \frac{1}{2} \eta_{\alpha\beta\gamma} F^{\beta\gamma}, \quad (11)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that from (11) it follows immediately that $g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi = 0$; so this case is related to, but different from, that of the wave equation (8) discussed previously. Since space is a sphere S^2 , there can be non-trivial topology: U(1) gauge fields over S^2 are classified topologically by the integer

$$k = \frac{-i}{2\pi} \int_\Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (12)$$

where Σ is a space section (spacelike surface with topology S^2).

An example of a topologically non-trivial solution of (11) is

$$\Phi = \frac{1}{2} i k (\cos \tilde{\theta} - 1), \quad A_\varphi = \frac{1}{2} i k (\cos \theta - 1), \quad A_\theta = 0 = A_{\tilde{\theta}}.$$

For smoothness, we need $A_\varphi = 0$ at $\theta = 0, \pi$; so the above gauge potential has a singularity at $\theta = \pi$. This is the familiar ‘Dirac-string’ singularity, and is removable: the gauge-transformed potential

$$A_\varphi + \exp(-ik\varphi) \partial_\varphi \exp(ik\varphi) = \frac{1}{2} i k (\cos \theta + 1)$$

is smooth near $\theta = \pi$. In other words, this Maxwell-Higgs system is smooth throughout M . The apparent singularities are a consequence of the fact that the gauge field is topologically non-trivial: its magnetic charge equals k . Furthermore, it is spatially-homogeneous: note in particular that Φ depends only on the time coordinate $\tilde{\theta}$, and that the gauge 2-form (the integrand of (12)) is a (time-dependent) multiple of the spatial area element $\sin \theta d\theta \wedge d\varphi$.

5. SPATIALLY-HOMOGENEOUS SU(2) SOLUTIONS.

Spatially-homogeneous SU(2) fields may be characterized as follows. Temporarily, think of the spatial 2-sphere as the unit sphere in \mathbf{R}^3 , with coordinates $x^j = (x^1, x^2, x^3)$. Take the Higgs field and gauge potential to have the form

$$\begin{aligned} \Phi &= i g(\tilde{\theta}) x^j \sigma_j, \\ A_j &= i f(\tilde{\theta}) \varepsilon_{jkl} x^l \sigma^k, \\ A_{\tilde{\theta}} &= 0, \quad (\text{a gauge choice}) \end{aligned} \quad (13)$$

where σ_j are the Pauli matrices, and f and g are two scalar functions of $\tilde{\theta}$ only. This implements SO(3) symmetry: recall, for example, that the spherically-symmetric 1-monopole in \mathbf{R}^3 has the ‘hedgehog’ form (13). The components A_θ and A_φ are obtained from A_j in the obvious way, by transforming to polar coordinates. Although Φ and A_μ depend on the spatial variables (θ, φ) , the effect of a spatial rotation is to make a gauge transformation; and gauge-invariant quantities such as $-\text{tr } \Phi^2 = 2g^2$ depend only on $\tilde{\theta}$.

Substituting (13) into (10) gives the pair of ordinary differential equations

$$\begin{aligned} g' &= 2f(1-f)\sin\tilde{\theta}, \\ f' &= g(2f-1)/\sin\tilde{\theta}. \end{aligned} \tag{14}$$

Eliminating g from these leaves an equation for f which, after the transformation

$$f(\tilde{\theta}) = \frac{1}{2}(e^{2T} + 1)P(T) + \frac{1}{2}, \quad \tanh T = -\cos\tilde{\theta},$$

is

$$P'' = (P')^2/P - 4e^{2T}P^3, \tag{15}$$

where $P' = dP/dT$. This is the third Painlevé equation P_{III} . In terms of the variable $t = e^T = \tan(\tilde{\theta}/2) \in (0, \infty)$, it takes the more usual form

$$\ddot{P} = (\dot{P})^2/P - \dot{P}/t - 4P^3, \tag{16}$$

where $\dot{P} = dP/dt$. Solutions of (15) or (16) therefore determine spatially-homogeneous SU(2) solutions of the Yang-Mills-Higgs-Bogomolny equations (10).

6. THE TWISTOR CORRESPONDENCE.

One can in principle construct all solutions of the self-dual Yang-Mills equations, and hence of (10), by using the twistor correspondence^{15,14}. The details of the construction are well-known, and here we simply give some brief details in order to establish notation and conventions.

Twistor space is the complex projective space \mathbf{CP}^3 , with homogeneous coordinates $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$. (Strictly speaking, the twistor space of \widetilde{M} is a non-Hausdorff space¹⁴ obtained by glueing together two copies of \mathbf{CP}^3 ; but for simplicity we shall avoid going into the details of this.) The correspondence between \mathbf{CP}^3 and \widetilde{M} is expressed by the relations

$$Z^0 = uZ^2 + wZ^3, \quad Z^1 = \bar{w}Z^2 + \bar{u}Z^3. \tag{17}$$

Here u and w are the complex coordinates defined by (2) (recall that they only cover ‘half’ of \widetilde{M} — it is for this reason that the true twistor space is a non-Hausdorff ‘doubling’ of \mathbf{CP}^3).

A matrix-valued twistor function $F(Z^\alpha)$ is said to be *real* if $F^\dagger = F$, where $F^\dagger(Z^\alpha) = F(\overline{Z^1}, \overline{Z^0}, \overline{Z^3}, \overline{Z^2})^*$, and $*$ denotes complex conjugate transpose. There is a correspondence between certain holomorphic vector bundles over twistor space, and solutions of the self-dual Yang-Mills equations on \widetilde{M} ; in particular, if $F(Z^\alpha)$ is a real ‘patching matrix’ for a vector bundle of rank n , then ‘splitting’ F yields a self-dual $U(n)$ gauge field. In addition to being real, the matrix function $F(Z^\alpha)$ has to be homogeneous of degree zero in Z^α ; and in order to have $\tilde{\varphi}$ -invariance, we require F to be annihilated by the vector field

$$V = Z^3 \frac{\partial}{\partial Z^3} - Z^2 \frac{\partial}{\partial Z^2} + Z^1 \frac{\partial}{\partial Z^1} - Z^0 \frac{\partial}{\partial Z^0}. \quad (18)$$

For example, all three requirements (reality, homogeneity and V -invariance) are met by the (scalar) function $Q = (Z^0 Z^1 + Z^2 Z^3)/(Z^2 Z^3)$. Indeed, the line bundle defined by the patching matrix $F = Q^k$, where k is an integer, yields the $U(1)$ solution of Section 4.

7. AN $SU(2)$ EXAMPLE.

In order to obtain $SU(2)$ solutions by this construction, we look for examples of 2×2 twistor matrices $F(Z^\alpha)$ which are upper-triangular, and which are equivalent to ‘real’ matrices. Given an upper-triangular F , one can obtain explicit expressions for Φ and A_μ (see, for example, section 8.2 of ref 15). The analogue of the ’tHooft ansatz, and its generalizations (corresponding to example 8.2.3 of ref 15) does not work — it produces only $SU(1,1)$ fields. But the analogue (changed-signature version) of example 8.2.4 of ref 15 does work, and produces $SU(2)$ solutions in our case. Some brief details are as follows.

Write $\zeta = Z^3/Z^2$, and think of $F(Z^\alpha)$ as defining a vector bundle by the patching relation $\hat{\psi} = F\psi$, where ψ and $\hat{\psi}$ are (2-vector) fibre-coordinates over $U = \{|\zeta| \leq 1\}$ and $\hat{U} = \{|\zeta| \geq 1\}$ respectively. Take $F(Z^\alpha)$ to have the form

$$F(Z^\alpha) = \begin{pmatrix} \zeta^k e^f & 2Q^{-1} \cosh f \\ 0 & \zeta^{-k} e^{-f} \end{pmatrix}, \quad (19)$$

where k is a positive integer, $f(Z^\alpha)$ is real, and $Q = P/(Z^2 Z^3)^k$ with $P(Z^\alpha)$ being a real polynomial (homogeneous of degree $2k$). Then, because

$$R(Z^\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & \zeta^k Q \end{pmatrix}$$

is holomorphic on U and FR is real, it follows that the construction will yield a real (*i.e.* $SU(2)$ -valued) solution.

As an example of this construction, take $P = (Z^0 Z^1 + Z^2 Z^3)$ and $k = 1$ (or $k = -1$, which leads to the same solution). The simplest choice for f , namely $f = 0$, gives nothing new: the field is then effectively abelian, and is an embedding into $SU(2)$ of the $U(1)$ solution described in section 4. To get something genuinely non-abelian, we may take $f = \log Q$, where $Q = (Z^0 Z^1 + Z^2 Z^3)/(Z^2 Z^3)$, so that

$$F(Z^\alpha) = \begin{pmatrix} \zeta Q & 1 + Q^{-2} \\ 0 & (\zeta Q)^{-1} \end{pmatrix}. \quad (20)$$

The procedure¹⁵ referred to above then yields explicit (although rather complicated) expressions for Φ and A_μ , as rational functions of $\cos \theta$, $\cos \tilde{\theta}$ and $\exp(i\varphi)$. The dependence on φ can be compensated by a gauge transformation, so in effect the solution depends only on θ and $\tilde{\theta}$: it is an $SO(2)$ -invariant solution of the Yang-Mills-Higgs equations (9) on M .

The functions are somewhat simpler when expressed in terms of the variables $X = \cos^2(\theta/2)$ and $Y = \cos^2(\tilde{\theta}/2)$; for example, $-\text{tr } \Phi^2 = \frac{1}{2}H(X, Y)/(1 + X^2 Y^2)^2$, where

$$H(X, Y) = 1 + 16X^4 Y^6 - 24X^4 Y^5 + 9X^4 Y^4 + 16X^2 Y^4 - 8X^2 Y^3 - 6X^2 Y^2 - 16XY^4 + 16XY^3.$$

Figure 1 contains plots of four gauge-invariant quantities, namely

$$\begin{aligned} K &:= -\text{tr } \Phi^2, \\ L &:= -\sin^2 \tilde{\theta} \text{tr}(D_{\tilde{\theta}} \Phi)^2, \\ M &:= -\sin^2 \tilde{\theta} \text{tr}[(D_\theta \Phi)^2 + (D_\varphi \Phi)^2 / \sin^2 \theta], \\ N &:= L - M = g^{\mu\nu} \text{tr}[(D_\mu \Phi)(D_\nu \Phi)], \end{aligned}$$

as functions of ‘spatial latitude’ X and ‘time’ Y . A couple of features that may be noted are:

- in the distant future or past (*i.e.* as $Y \rightarrow 1$ or $Y \rightarrow 0$), the field approaches a ‘vacuum value’ where $-\text{tr } \Phi^2 = \frac{1}{2}$ and $-\text{tr}(D_\mu \Phi)^2 = 0$;
- at the point $X = 0$ on the spatial sphere, we have $-\text{tr } \Phi^2 = \frac{1}{2}$, $-\text{tr}(D_{\text{time}} \Phi)^2 = 0$ and $-\text{tr}(D_{\text{space}} \Phi)^2 = 16Y^4(Y - 1)^2$.

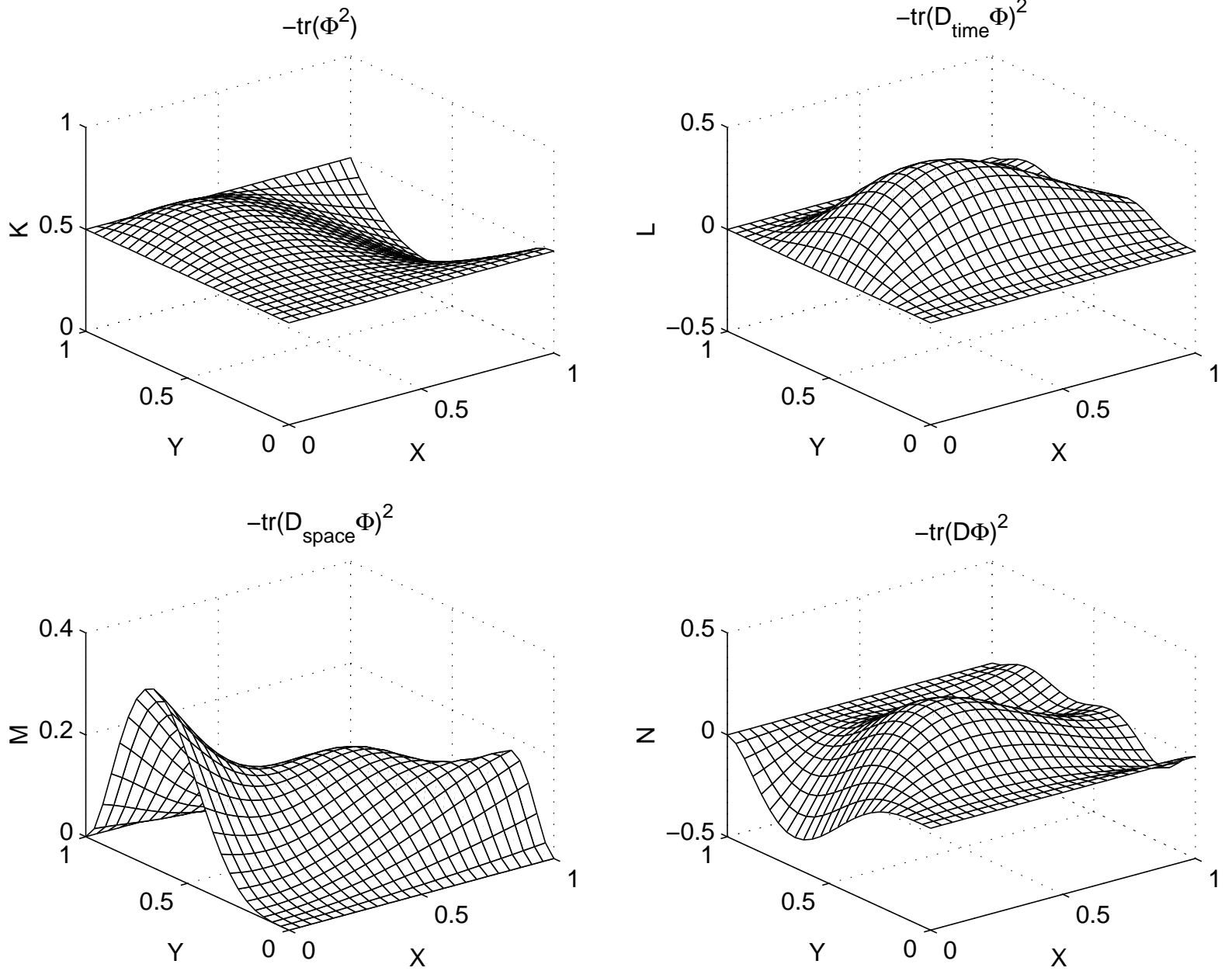


Fig. 1. The quantities $K = -\text{tr } \Phi^2$, $L = -\sin^2 \tilde{\theta} \text{tr}(D_{\tilde{\theta}} \Phi)^2$, $M = -\sin^2 \tilde{\theta} \text{tr}[(D_{\theta} \Phi)^2 + (D_{\varphi} \Phi)^2 / \sin^2 \theta]$ and $N = -g^{\mu\nu} \text{tr}[(D_{\mu} \Phi)(D_{\nu} \Phi)]$ as functions of $X = \cos^2(\theta/2)$ and $Y = \cos^2(\tilde{\theta}/2)$.

8. CONCLUDING REMARKS.

For the corresponding systems in (2+1)-dimensional flat² and anti-de Sitter¹¹ space-time, there are localized soliton solutions; and a single soliton travels (as one would expect) along a timelike geodesic. More investigation is needed to determine whether the same is

true in the de Sitter case. The method used to construct solutions in the former cases does not work so well here; the construction of section 7 is, by contrast, the analogue of one which yields the one-monopole solution¹⁵ of the Yang-Mills-Higgs-Bogomolnyi equations on \mathbf{R}^3 . One question, therefore, is whether there is a meaningful correspondence between these two systems, *i.e.* between the Yang-Mills-Higgs systems on \mathbf{R}^3 and on (2+1)-dimensional de Sitter space.

In addition to exact solution methods, one may wish to investigate the equations numerically, as was done in the flat case⁵. For this, an alternative sigma-model or chiral-model formulation is useful; and this may be of interest in any event. For example, there exists a gauge in which $A_{\bar{u}} = H^{-1}\partial_{\bar{u}}H$ and $A_w = H^{-1}\partial_wH$, where H takes values in the complexified gauge group (*i.e.* $SL(2, \mathbb{C})$ if $G = su(2)$). Then the hermitian matrix $K = HH^*$ satisfies

$$\partial_u(K^{-1}\partial_{\bar{u}}K) - \partial_{\bar{w}}(K^{-1}\partial_wK) = 0. \quad (21)$$

And this single matrix equation (21) is equivalent, after transforming coordinates as in (2) and imposing a suitable dependence on $\tilde{\varphi}$, to the Yang-Mills-Higgs equations (10).

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